## Problem 1. Second quantization

In this question $c_i^\dagger$ and $c_i$ are fermion creation and annihilation operators and the states are fermion states. Use the convention $|1111100\cdots\rangle = c_5^\dagger c_4^\dagger c_3^\dagger c_2^\dagger c_1^\dagger |000\cdots\rangle$.

(a) Use the anticommutation relations for fermions to "normal-order" $c_3^\dagger c_6 c_4 c_6^\dagger c_3$ ("normal-order" means commuting all annihilation operators to the right, so for instance $c_2 c_1 c_1^\dagger$ normal ordered would be $c_2 c_1 c_1^\dagger = -c_1^\dagger c_1 c_2 + c_2$).

(b) Evaluate $c_3^\dagger c_6 c_4 c_6^\dagger c_3 |111111000\cdots\rangle$ and $c_3^\dagger c_6 c_4 c_6^\dagger c_3 |111110000\cdots\rangle$.

(c) Write $|1101100100\cdots\rangle$ in terms of excitations about the "filled Fermi sea" $|\Omega\rangle = |1111100000\cdots\rangle$. Interpret your answer in terms of electron and hole excitations.

(d) Find $\langle\psi|\hat{N}|\psi\rangle$, where $|\psi\rangle = A|100000\rangle + B|111000\rangle$, $\hat{N} = \sum_i c_i^\dagger c_i$.

## Problem 2. Bogoliubov transformations

Consider two fermions $a_1$ and $a_2$

(a) Show that the Bogoliubov transformation

$$\begin{aligned} c_1 &= u a_1 + v a_2^\dagger, \\ c_2^\dagger &= -v a_1 + u a_2^\dagger. \end{aligned} \tag{1}$$

where $u$ and $v$ are real, preserves the canonical anticommutation relations if $u^2+v^2=1$.

(b) Use this result to show that the Hamiltonian

$$H = \epsilon(a_1^\dagger a_1 - a_2 a_2^\dagger) + \Delta(a_1^\dagger a_2^\dagger + \text{h.c.}), \tag{2}$$

can be diagonalized in the form

$$H = \sqrt{\epsilon^2 + \Delta^2}\,(c_1^\dagger c_1 + c_2^\dagger c_2 - 1). \tag{3}$$

(c) What is the ground-state energy of this Hamiltonian?

(d) Write out the ground-state wavefunction in terms of the original operators $a_1^\dagger$ and $a_2^\dagger$ and their corresponding vacuum $|0\rangle$ (i.e., $a_{1,2}|0\rangle = 0$).

## Problem 3. Self-consistency in BCS superconductivity

In class we derived the following Hamiltonian (which was valid for $\mathbf{k}$ near the Fermi surface)

$$H = \sum_{\mathbf{k},\sigma} (\epsilon_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} - \frac{g}{V} \sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow} c_{-\mathbf{k}',\downarrow} c_{\mathbf{k}',\uparrow}. \tag{4}$$

(a) Repeat the mean-field ansatz $\Delta = -\frac{g}{V} \langle \sum_{\mathbf{k}} c_{-\mathbf{k}',\downarrow} c_{\mathbf{k}',\uparrow} \rangle$ to obtain the mean-field Hamiltonian

$$H_{\text{MFT}} = \sum_{\mathbf{k},\sigma} (\epsilon_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} + \sum_{\mathbf{k}} [\Delta^* c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} + \text{h.c.}] + \frac{V}{g} |\Delta|^2 \tag{5}$$

(b) With what we learned in Problem 2, make a Bogoliubov transformation to put this Hamiltonian into the form

$$H_{\text{MFT}} = \sum_{\mathbf{k}} E_{\mathbf{k}} (a^\dagger_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma} - 1/2) + \frac{V}{g} |\Delta|^2. \tag{6}$$

(c) What is the ground-state wave-function for this system (write in terms of the electron vacuum $|0\rangle$ and $c_{\mathbf{k},\sigma}$ operators)? (Hint: Given the state $|0\rangle$ annihilated by $c$ operators, the state $a_{-\mathbf{k},\uparrow} a_{\mathbf{k},\downarrow} |0\rangle$ is annihilated by $a_{-\mathbf{k},\uparrow}$ and $a_{\mathbf{k},\downarrow}$.)

(d) Call the wave function from the previous part $|\psi_{\text{BCS}}\rangle$. Show that the self-consistent equation derived from $\Delta = -\frac{g}{V} \langle \sum_{\mathbf{k}} c_{-\mathbf{k}',\downarrow} c_{\mathbf{k}',\uparrow} \rangle$ is

$$\Delta = g \int_{|\epsilon_{\mathbf{k}} - \mu| < \omega_D} \frac{d^3k}{(2\pi)^3} \frac{\Delta}{2\sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta^2}}. \tag{7}$$

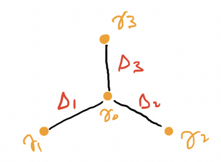
(e) Finally, find a nonzero approximate solution to (d) in terms of terms of the $g$, $\omega_D$, and the density of states at the Fermi level $\rho_0$. (Approximations are needed, if you get stuck, look up in a book that covers superconductivity).

## Problem 4. Braiding Majoranas

Consider the following Hamiltonian for 4 Majorana fermions

$$H = i \sum_{i=1}^{3} \Delta_i \gamma_0 \gamma_i. \tag{8}$$

This can be made by taking three wires in the following geometry and tuning the superconducting gap between neighboring pairs.




In this problem, we are using $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$.

(a) Put the Hamiltonian in block-diagonal form with $\tilde{\gamma}_\mu = \sum_\nu O_{\mu\nu}\gamma_\nu$ such that

$$H = \frac{i}{2}\tilde{\gamma}^T \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix} \tilde{\gamma} \tag{9}$$

What are $\epsilon_1$ and $\epsilon_2$? (ensure $\tilde{\gamma}$ are properly normalized)

(b) Define two fermions $c_1 = \frac{1}{2}(\gamma_1 - i\gamma_2)$ and $c_2 = \frac{1}{2}(\gamma_0 - i\gamma_3)$, and define a basis for the Hilbert space as $|11\rangle = c_2^\dagger c_1^\dagger |0\rangle$, $|10\rangle = c_1^\dagger |0\rangle$, $|01\rangle = c_2^\dagger |0\rangle$, and $c_1 |0\rangle = 0 = c_2 |0\rangle$. What is the Hamiltonian $H$ in this basis? *Hint*: It will be in the form:

$$H = \begin{bmatrix} H_{\text{even}} & 0, \\ 0 & H_{\text{odd}}. \end{bmatrix} \tag{10}$$

where the rows are given by $|00\rangle$, $|11\rangle$, $|01\rangle$, and $|10\rangle$ with $H_{\text{even}}$ and $H_{\text{odd}}$ two-by-two matrices.

(c) Note that the parity operator $P = \gamma_0\gamma_1\gamma_2\gamma_3$ has eigenvalue $+1$ for $|00\rangle$ and $|11\rangle$ and $-1$ for $|01\rangle$ and $|10\rangle$. When $\Delta_1 = 0 = \Delta_2$ what is the ground state manifold of $H$? Show that when we restrict to the ground state manifold that $P' = i\gamma_1\gamma_2$ acts the same as $P$.

(d) For $H_{\text{even}}$ find the ground states when (1) $\Delta_{1,2} = 0$, (2) $\Delta_{2,3} = 0$, and (3) $\Delta_{1,3} = 0$. Compute the Berry phase for the path (1) $\to$ (2) $\to$ (3) $\to$ (1).

(e) Repeat (d) for $H_{\text{odd}}$.

(f) Using the ground states and operator in (c), create a unitary $U = e^{i\phi P'}$ that changes each state by the Berry phase computed in (d,e).

(g) Compute $U\gamma_1 U^\dagger$ and $U\gamma_2 U^\dagger$ to show that these Majorana fermions were exchanged – we have braided two Majoranas.